



## § 4.3 分部积分法

**微分运算**中有两个重要法则：  
复合函数微分法和乘积的微分法。  
在**积分运算**中，与它们对应的是上节的  
换元积分法和本节的部分积分法——  
基本积分法**(两种)**。



## •分部积分公式

设函数 $u=u(x)$ 及 $v=v(x)$ 具有连续导数. 那么,

$$(uv)'=u'v+uv',$$

移项得

$$uv'=(uv)'-u'v.$$

对这个等式两边求不定积分, 得

$$\int uv' dx = uv - \int u'v dx, \quad \text{或} \quad \int u dv = uv - \int v du,$$

这两个公式称为分部积分公式.

## •分部积分过程

$$\int uv' dx = \int u dv = uv - \int v du = uv - \int u'v dx = \dots$$



分部积分过程:  $\int u v' dx = \int u dv = uv - \int v du = uv - \int v u' dx$

例1 求  $\int x \cos x dx$ .

解  $\int x \cos x dx = \int x d(\sin x)$

$$= x \sin x - \int \sin x dx$$

降  
幂

$$= x \sin x + \cos x + C$$

讨论:  $\int x \cos x dx = \int \cos x d\left(\frac{x^2}{2}\right) = \frac{x^2}{2} \cos x - \int \frac{x^2}{2} d \cos x$

$$= \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x dx$$

升  
幂



分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例2 求  $\int x^2 e^x dx$ .

解  $\int x^2 e^x dx = \int x^2 de^x$

$$= x^2 e^x - \int e^x dx^2 = x^2 e^x - 2 \int x e^x dx$$

降  
幂

$$= x^2 e^x - 2 \int x de^x = x^2 e^x - 2(xe^x - \int e^x dx)$$

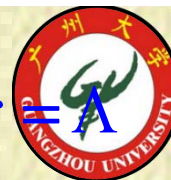
降  
幂

$$= x^2 e^x - 2(xe^x - e^x) + C$$

讨论  $\int x^2 e^x dx = \int e^x d(\frac{x^3}{3})$

$$= \frac{x^3 e^x}{3} - \frac{1}{3} \int x^3 e^x dx$$

升  
幂

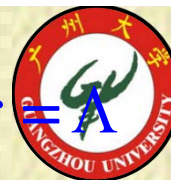


分部积分过程： $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

$$\int P_n(x) \sin ax dx, \int P_n(x) \cos ax dx, \int P_n(x) e^{kx} dx,$$

其中  $k, a$  为常数,  $P_n(x)$  为  $n$  次多项式

用分部积分法, 使多项式的次数降低



分部积分过程： $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例3  $\int x \ln x dx$   $x dx = \frac{1}{2} dx^2$

$$= \frac{1}{2} \int \ln x dx^2 = \frac{1}{2} \left( x^2 \ln x - \int x^2 d \ln x \right)$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \cdot \frac{1}{x} dx$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx$$

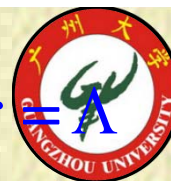
$$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C .$$



分部积分过程:  $\int u v' dx = \int u dv = uv - \int v du = uv - \int v u' dx$

例4  $\int \arccos x dx$

$$= x \arccos x - \int x d \arccos x$$
$$= x \arccos x + \int x \frac{1}{\sqrt{1-x^2}} dx$$
$$= x \arccos x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(1-x^2)$$
$$= x \arccos x - \sqrt{1-x^2} + C .$$



分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例5  $\int x \arctan x dx$   $x dx = \frac{1}{2} dx^2$

$$= \frac{1}{2} \int \arctan x dx^2$$
$$= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int x^2 \cdot \frac{1}{1+x^2} dx$$
$$= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$$
$$= \frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C .$$



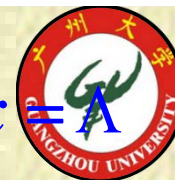


分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例6 求  $\int \frac{\ln(x-1)}{x^2} dx$ .  $\frac{1}{x^2} dx = d(-\frac{1}{x})$

解

$$\begin{aligned} & \int \frac{\ln(x-1)}{x^2} dx \\ &= \int \ln(x-1) d(-\frac{1}{x}) \\ &= -\frac{\ln(x-1)}{x} + \int \frac{1}{x} \cdot \frac{1}{x-1} dx \\ &= -\frac{\ln(x-1)}{x} + \int (\frac{1}{x-1} - \frac{1}{x}) dx \\ &= -\frac{\ln(x-1)}{x} + \ln(x-1) - \ln x + C. \end{aligned}$$



分部积分过程： $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

$$\int x^n \arcsin x dx, \quad \int x^n \arctan x dx,$$
$$\int x^n \ln P(x) dx \quad (n \neq -1)$$

用分部积分法，去掉反三角函数、对数



分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例7 求  $\int e^x \sin x dx$ .

解  $\int e^x \sin x dx = \int \sin x de^x$

$$= e^x \sin x - \int e^x d(\sin x)$$

$$= e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x de^x$$

$$= e^x \sin x - (e^x \cos x - \int e^x d \cos x)$$

$$= e^x (\sin x - \cos x) - \int e^x \sin x dx$$

原积分回归

$$\int e^x \sin x dx = \frac{e^x}{2} (\sin x - \cos x) + C.$$



分部积分过程： $\int u v' dx = \int u dv = uv - \int v du = uv - \int v u' dx$

$$\int e^{kx} \sin(ax + b) dx, \int e^{kx} \cos(ax + b) dx,$$

其中 $k, a, b$ 均为常数

$u, dv$ 的选取可随意

注意前后几次所选的 $u$ 应为同类型函数

用分部积分法, 建立回归方程



分部积分过程:  $\int u v' dx = \int u dv = uv - \int v du = uv - \int v u' dx$

例8 求  $\int \sec^3 x dx$  .

解  $\int \sec^3 x dx = \int \sec x d \tan x$

$$= \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$= \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x dx \quad \text{回归}$$

所以  $\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$  .



分部积分过程:  $\int u v' dx = \int u dv = uv - \int v du = uv - \int v u' dx$

例9 推导以下递推公式:

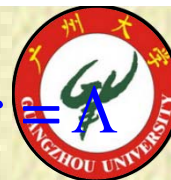
$$\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

解

$$\begin{aligned} \int \cos^n x dx &= \int \cos^{n-1} x d \sin x \\ &= \sin x \cos^{n-1} x - \int \sin x \cdot (n-1) \cos^{n-2} x \cdot (-\sin x) dx \\ &= \sin x \cos^{n-1} x + (n-1) \left( \int \cos^{n-2} x dx - \int \cos^n x dx \right), \end{aligned}$$

回归

$$\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx.$$



分部积分过程： $\int uv'dx = \int u dv = uv - \int v du = uv - \int vu'dx$

### 分部积分基本题型：

1)  $\int P(x)\sin(ax+b)dx$  ,  $\int P(x)\cos(ax+b)dx$  ,  
 $\int P(x)e^{ax}dx$  .....

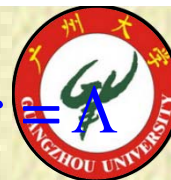
取  $u = P(x)$

2)  $\int x^n \ln P(x)dx$  ,  $\int x^n \arctan x dx$  .....

取  $dv = x^n dx$

3)  $\int e^{ax} \sin bxdx$  ,  $\int e^{ax} \cos bxdx$  ,  $\int \sec^3 x dx$  .....

分部积分“回归法”



分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例10 求  $\int 2x \ln(x^2 + 1) dx$ .

解

$$\begin{aligned} & \int 2x \ln(x^2 + 1) dx \\ &= \int \ln(x^2 + 1) d(x^2 + 1) \\ &= (x^2 + 1) \ln(x^2 + 1) - \int (x^2 + 1) \cdot \frac{2x}{x^2 + 1} dx \\ &= (x^2 + 1) \ln(x^2 + 1) - \int 2x dx \\ &= (x^2 + 1) \ln(x^2 + 1) - x^2 + C. \end{aligned}$$





分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例11 求  $\int e^{\sqrt{2x+1}} dx$ .

解 令  $t = \sqrt{2x+1}$ , 则

$$x = \frac{1}{2}(t^2 - 1), \quad dx = t dt,$$

$$\int e^{\sqrt{2x+1}} dx = \int e^t \cdot t dt = \int t de^t$$

$$= te^t - \int e^t dt = te^t - e^t + C$$

$$= (\sqrt{2x+1} - 1)e^{\sqrt{2x+1}} + C.$$



分部积分过程:  $\int uv' dx = \int u dv = uv - \int v du = uv - \int vu' dx$

例12 求  $I_n = \int \frac{dx}{(x^2 + a^2)^n}$ , 其中  $n$  为正整数.

解  $I_1 = \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$ ;

当  $n > 1$  时, 用分部积分法, 有

$$\begin{aligned} \int \frac{dx}{(x^2 + a^2)^{n-1}} &= \frac{x}{(x^2 + a^2)^{n-1}} + 2(n-1) \int \frac{x^2}{(x^2 + a^2)^n} dx \\ &= \frac{x}{(x^2 + a^2)^{n-1}} + 2(n-1) \int \left[ \frac{1}{(x^2 + a^2)^{n-1}} - \frac{a^2}{(x^2 + a^2)^n} \right] dx, \end{aligned}$$

回归

即 
$$I_{n-1} = \frac{x}{(x^2 + a^2)^{n-1}} + 2(n-1)(I_{n-1} - a^2 I_n),$$

于是 
$$I_n = \frac{1}{2a^2(n-1)} \left[ \frac{x}{(x^2 + a^2)^{n-1}} + (2n-3)I_{n-1} \right].$$



## 思考题1

已知  $f(x)$  的一个原函数是  $e^{-x^2}$ ，求  $\int xf'(x)dx$ 。

**解** 
$$\int x f'(x)dx = \int xdf(x) = xf(x) - \int f(x)dx,$$

$$\ominus \int f(x)dx = e^{-x^2} + C$$

两边同时对  $x$  求导，得  $f(x) = -2xe^{-x^2}$

$$\begin{aligned} \therefore \int xf'(x)dx &= xf(x) - \int f(x)dx \\ &= -2x^2e^{-x^2} - e^{-x^2} + C \end{aligned}$$



## 思考题2

求积分  $\int \sin(\ln x) dx$ .

解  $\int \sin(\ln x) dx = x \sin(\ln x) - \int x d[\sin(\ln x)]$

$$= x \sin(\ln x) - \int x \cos(\ln x) \cdot \frac{1}{x} dx$$

$$= x \sin(\ln x) - x \cos(\ln x) + \int x d[\cos(\ln x)]$$

$$= x[\sin(\ln x) - \cos(\ln x)] - \int \sin(\ln x) dx$$

回归

$$\therefore \int \sin(\ln x) dx = \frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C.$$



### 思考题3

求积分  $\int \frac{x \arctan x}{\sqrt{1+x^2}} dx$ .

$$\frac{x}{\sqrt{1+x^2}} dx = d\sqrt{1+x^2}$$

**解**

$$\begin{aligned} \int \frac{x \arctan x}{\sqrt{1+x^2}} dx &= \int \arctan x d\sqrt{1+x^2} \\ &= \sqrt{1+x^2} \arctan x - \int \sqrt{1+x^2} d(\arctan x) \\ &= \sqrt{1+x^2} \arctan x - \int \sqrt{1+x^2} \cdot \frac{1}{1+x^2} dx \\ &= \sqrt{1+x^2} \arctan x - \int \frac{1}{\sqrt{1+x^2}} dx \\ &= \sqrt{1+x^2} \arctan x - \ln(x + \sqrt{1+x^2}) + C. \end{aligned}$$



# 作业

习题4-3 (P210):

3. 5. 7. 9. 10.  
15. 16. 19. 20. 21.