§4.3 分部积分法

微分运算中有两个重要法则:

复合函数微分法和乘积的微分法.

在积分运算中, 与它们对应的是上节的换元积分法和本节的分部积分法 ——

基本积分法(两种).

•分部积分公式

设函数$u=u(x)$及$v=v(x)$具有连续导数. 那么,

$$(uv)'=u'v+uv',$$

移项得

$$uv'=(uv)'-u'v.$$

对这个等式两边求不定积分, 得

$$\int uv'dx=uv-\int u'vdx,\quad \text{或}\int udv=uv-\int vdu,$$

这两个公式称为分部积分公式.

•分部积分过程

$$\int uv'dx=\int udv=uv-\int vdu=uv-\int u'vdx=\cdots.$$

分部积分过程：$\int uv'\mathrm{d}x = \boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u} = uv - \int vu'\,\mathrm{d}x = \Lambda$

例1 求$\int x\cos x\mathrm{d}x.$

解 $\int x\cos x\mathrm{d}x = \int x\mathrm{d}(\sin x)$

$$= x\sin x - \boxed{\int \sin x\mathrm{d}x}$$

降幂

$$= x\sin x + \cos x + C$$

讨论: $\int x\cos x\mathrm{d}x = \int \cos x\mathrm{d}(\frac{x^2}{2}) = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}\mathrm{d}\cos x$

$$= \frac{x^2}{2}\cos x + \boxed{\int \frac{x^2}{2}\sin x\mathrm{d}x}$$

升幂

例2 求 $\int x^2 e^x \mathrm{d}x.$

解 $\int x^2 e^x \mathrm{d}x = \int x^2 \mathrm{d}e^x$

$= x^2 e^x - \int e^x \mathrm{d}x^2 = x^2 e^x - 2\int xe^x \mathrm{d}x$ 降幂

$= x^2 e^x - 2\int x\mathrm{d}e^x = x^2 e^x - 2(xe^x - \int e^x \mathrm{d}x)$ 降幂

$= x^2 e^x - 2(xe^x - e^x) + C$

讨论 $\int x^2 e^x \mathrm{d}x = \int e^x \mathrm{d}(\frac{x^3}{3})$

$= \frac{x^3 e^x}{3} - \frac{1}{3}\int x^3 e^x \mathrm{d}x$ 升幂

分部积分过程： $\int uv'dx = \int u\,dv = uv - \int v\,du = uv - \int vu'\,dx = \Lambda$

u u u

$\int P_n(x)\sin ax\,dx, \int P_n(x)\cos ax\,dx, \int P_n(x)e^{kx}dx,$

其中k, a为常数, $P_n(x)$为n次多项式

用分部积分法，使多项式的次数降低

例3 $\int x\ln x dx$

$$xdx = \frac{1}{2}dx^2$$

$$= \frac{1}{2}\int \ln x dx^2 = \frac{1}{2}\left(x^2\ln x - \int x^2 d\ln x\right)$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

例4 $\int \arccos x dx$

$$= x\arccos x - \int x d\arccos x$$

$$= x\arccos x + \int x\frac{1}{\sqrt{1-x^2}}dx$$

$$= x\arccos x - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d(1-x^2)$$

$$= x\arccos x - \sqrt{1-x^2} + C\ .$$

例5 $\int x\arctan x dx$

$$x\mathrm{d}x = \frac{1}{2}\mathrm{d}x^2$$

$$= \frac{1}{2}\int \arctan x dx^2$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int x^2\cdot\frac{1}{1+x^2}dx$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int(1-\frac{1}{1+x^2})dx$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C\,.$$

例6 求 $\int \frac{\ln(x-1)}{x^2}\mathrm{d}x.$

$$\frac{1}{x^2}\mathrm{d}x = \mathrm{d}(-\frac{1}{x})$$

解

$$\int \frac{\ln(x-1)}{x^2}\mathrm{d}x$$

$$= \int \ln(x-1)\mathrm{d}(-\frac{1}{x})$$

$$= -\frac{\ln(x-1)}{x} + \int \frac{1}{x}\cdot\frac{1}{x-1}\mathrm{d}x$$

$$= -\frac{\ln(x-1)}{x} + \int (\frac{1}{x-1} - \frac{1}{x})\mathrm{d}x$$

$$= -\frac{\ln(x-1)}{x} + \ln(x-1) - \ln x + C.$$

分部积分过程： $\int uv'\mathrm{d}x = \boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u} = uv - \int vu'\,\mathrm{d}x = \Lambda$

$$\int x^n \underbrace{\arcsin x}_{u}\,\mathrm{d}x,\quad \int x^n \underbrace{\arctan x}_{u}\,\mathrm{d}x,$$

$$\int x^n \underbrace{\ln P(x)}_{u}\,\mathrm{d}x \qquad (n \neq -1)$$

用分部积分法，去掉反三角函数、对数

例7　求 $\int e^x \sin x\mathrm{d}x.$

解　$\int e^x \sin x\mathrm{d}x = \int \sin x\mathrm{d}e^x$

$= e^x \sin x - \int e^x \mathrm{d}(\sin x)$

$= e^x \sin x - \int e^x \cos x\mathrm{d}x = e^x \sin x - \int \cos x\mathrm{d}e^x$

$= e^x \sin x - (e^x \cos x - \int e^x \mathrm{d}\cos x)$

$= e^x(\sin x - \cos x) - \int e^x \sin x\mathrm{d}x$　　原积分回归

$$\int e^x \sin x\mathrm{d}x = \frac{e^x}{2}(\sin x - \cos x) + C.$$

$\int e^{kx}\sin(ax+b)\mathrm{d}x,\ \int e^{kx}\cos(ax+b)\mathrm{d}x,$

其中k, a, b均为常数

$u, \mathrm{d}v$的选取可随意

注意前后几次所选的 u 应为同类型函数

用分部积分法，建立回归方程

例8 求 $\int \sec^3 x dx$.

解 $\int \sec^3 x dx = \int \sec x d\tan x$

$= \sec x \tan x - \int \sec x \tan^2 x dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$

$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$

$= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x dx$, 回归

所以 $\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$.

例9 推导以下递推公式:

$$\int \cos^n x\,\mathrm{d}x = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,\mathrm{d}x.$$

解 $\boxed{\int \cos^n x\,\mathrm{d}x} = \int \cos^{n-1}x\,\mathrm{d}\sin x$

$$= \sin x\cos^{n-1}x - \int \sin x\cdot(n-1)\cos^{n-2}x\cdot(-\sin x)\mathrm{d}x$$

$$= \sin x\cos^{n-1}x + (n-1)\left(\int \cos^{n-2}x\,\mathrm{d}x - \boxed{\int \cos^n x\,\mathrm{d}x}\right),$$

回归

$$\int \cos^n x\,\mathrm{d}x = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,\mathrm{d}x.$$

分部积分过程：$\int uv'\mathrm{d}x = \boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u} = uv - \int vu'\,\mathrm{d}x = \Lambda$

分部积分基本题型：

1） $\int P(x)\sin(ax+b)\mathrm{d}x$, $\int P(x)\cos(ax+b)\mathrm{d}x$,

$\int P(x)e^{ax}\mathrm{d}x$ ……

取 $u = P(x)$

2） $\int x^n \ln P(x)\mathrm{d}x$, $\int x^n \arctan x\mathrm{d}x$ ……

取 $\mathrm{d}v = x^n\mathrm{d}x$

3） $\int e^{ax}\sin bx\mathrm{d}x, \int e^{ax}\cos bx\mathrm{d}x$, $\int \sec^3 x\mathrm{d}x$ ……

分部积分“回归法”

例10 求 $\int 2x\ln(x^2+1)\mathrm{d}x.$

解 $\int 2x\ln(x^2+1)\mathrm{d}x$

$$= \int \ln(x^2+1)\mathrm{d}(x^2+1)$$

$$= (x^2+1)\ln(x^2+1) - \int (x^2+1)\cdot\frac{2x}{x^2+1}\mathrm{d}x$$

$$= (x^2+1)\ln(x^2+1) - \int 2x\mathrm{d}x$$

$$= (x^2+1)\ln(x^2+1) - x^2 + C.$$

例11 求 $\int e^{\sqrt{2x+1}}\,\mathrm{d}x.$

解 令 $t = \sqrt{2x+1}$, 则

$$x = \frac{1}{2}(t^2 - 1), \quad \mathrm{d}x = t\mathrm{d}t,$$

$$\int e^{\sqrt{2x+1}}\,\mathrm{d}x = \int e^t \cdot t\,\mathrm{d}t = \int t\,\mathrm{d}e^t$$

$$= te^t - \int e^t\,\mathrm{d}t = te^t - e^t + C$$

$$= (\sqrt{2x+1} - 1)e^{\sqrt{2x+1}} + C.$$

例12 求 $I_n = \int \frac{dx}{(x^2+a^2)^n}$, 其中 n 为正整数.

解 $I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$;

当$n>1$时, 用分部积分法, 有

$$\int \frac{dx}{(x^2+a^2)^{n-1}} = \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{x^2}{(x^2+a^2)^n}dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int\left[\frac{1}{(x^2+a^2)^{n-1}} - \frac{a^2}{(x^2+a^2)^n}\right]dx,$$

回归

即 $$I_{n-1} = \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)(I_{n-1} - a^2 I_n),$$

于是 $$I_n = \frac{1}{2a^2(n-1)}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)I_{n-1}\right].$$

思考题1

已知 $f(x)$ 的一个原函数是 e^{-x^2}，求 $\int xf'(x)dx$.

解 $\int x\, f'(x)\mathrm{d}x = \int x\mathrm{d}f(x) = xf(x) - \int f(x)dx,$

$\Theta \quad \int f(x)\mathrm{d}x = e^{-x^2} + C$

两边同时对x求导，得 $f(x) = -2xe^{-x^2}$

$\therefore \quad \int xf'(x)\mathrm{d}x = xf(x) - \int f(x)\mathrm{d}x$

$= -2x^2e^{-x^2} - e^{-x^2} + C$

思考题2

求积分$\int \sin(\ln x)dx$.

解 $\int \sin(\ln x)dx = x\sin(\ln x) - \int x d[\sin(\ln x)]$

$$= x\sin(\ln x) - \int x\cos(\ln x)\cdot\frac{1}{x}dx$$

$$= x\sin(\ln x) - x\cos(\ln x) + \int x d[\cos(\ln x)]$$

$$= x[\sin(\ln x) - \cos(\ln x)] - \int \sin(\ln x)dx$$

回归

$$\therefore \int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

思考题3

求积分 $\int \frac{x\arctan x}{\sqrt{1+x^2}}dx.$

$$\frac{x}{\sqrt{1+x^2}}dx = d\sqrt{1+x^2}$$

解

$$\int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \arctan x d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2}\, d(\arctan x)$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2} \cdot \frac{1}{1+x^2}dx$$

$$= \sqrt{1+x^2}\arctan x - \int \frac{1}{\sqrt{1+x^2}}dx$$

$$= \sqrt{1+x^2}\arctan x - \ln(x+\sqrt{1+x^2}) + C.$$

作 业

习题4–3 (P210):

3.　5.　7.　9.　10.

15.　16.　19.　20.　21.